Unit No: H7K1 34

Unit Title: Eng Maths 2

1.2 Applications of Differentiation

Maxima and Minima

Graph Sketching

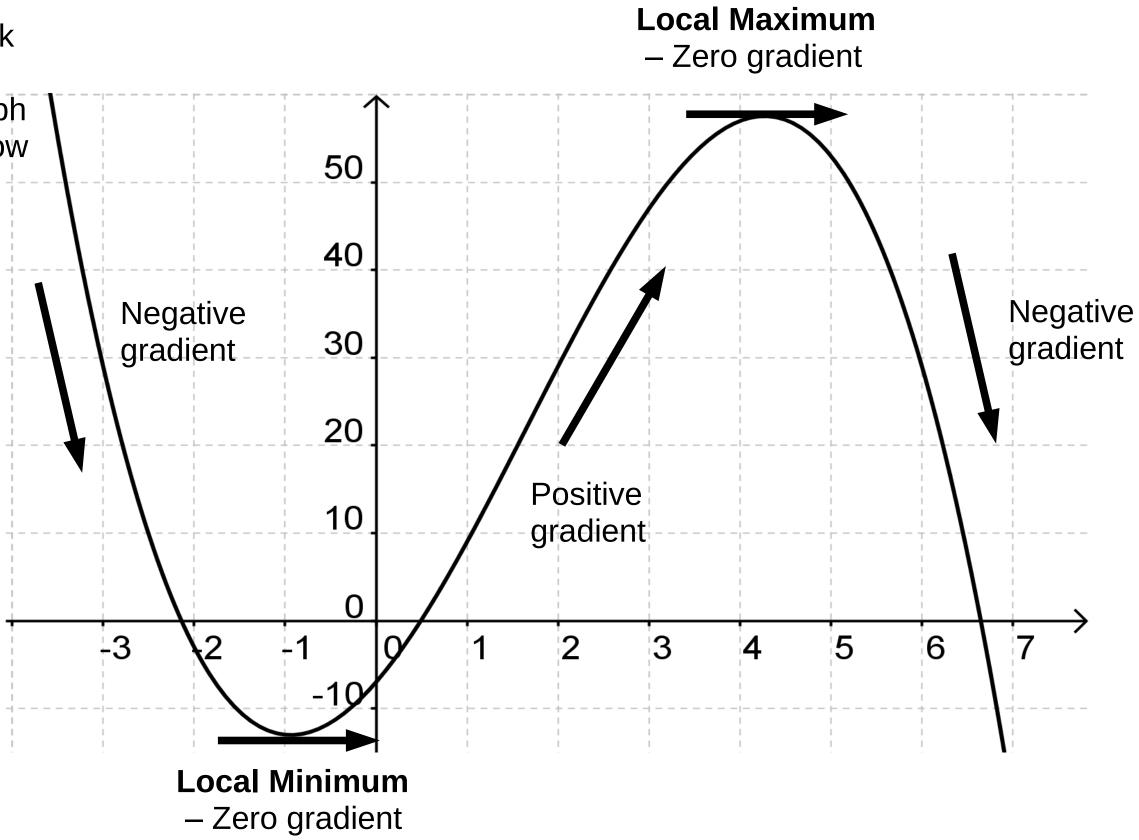
Practical Problems

19/11/20

Engineering Science and Technology

Maxima and Minima

Look at the graph below



The local maximum is often called a “maximum turning point” and the local minimum the “minimum turning point”.

What is the gradient of the line at the turning points? ___________

At a turning point
the differential of the equation of the line will be equal to _________

Imagine how you might be able to sketch the curve above if you only knew the equation. Move on to page 2.

I.e. The equation of the graph above is

$$y = 12x - 7 + 5x^2 - x^3 \qquad \text{Eq 1}$$

Its differential will be

$$\frac{dy}{dx} = 12 + 10x - 3x^2 \qquad \text{Eq 2}$$

At a local maximum or local minimum $\frac{dy}{dx} = 0$ (or at a turning point $\frac{dy}{dx} = 0$)

Therefore $12 + 10x - 3x^2 = 0$

The x values that make $12 + 10x - 3x^2$ equal to zero will be the values of the x (horizontal) co-ordinates of the local maximum and local minimum.

How do we find these x values? Use ________________________________.

I obtained 2 values $x = -0.9368$, $x = 4.270$ both to 4sf

Make sure that you can find these values too.

You must check that these are, in fact, values of x which give the maximum and minimum and you must check which is which. This can be done by looking at the gradient on each side of your possible maximum and minimum. A table is required.

Please check that the answers are correct

The values of x are ordered as they would be on the horizontal axis. Use Eq 2

	Left		Right		Left		Right
x	-0.9400	-0.9368	-0.9300		4.2600	4.2700	4.2800
$\frac{dy}{dx}$	-0.0508	-0.0008 ≈ 0	0.1053		0.1572	0.0013 ≈ 0	-0.1552
Gradient	↘	→	↗		↗	→	↘

Note: I am taking the x values of -0.0008 and 0.0013 to be 0.00 (rounded).

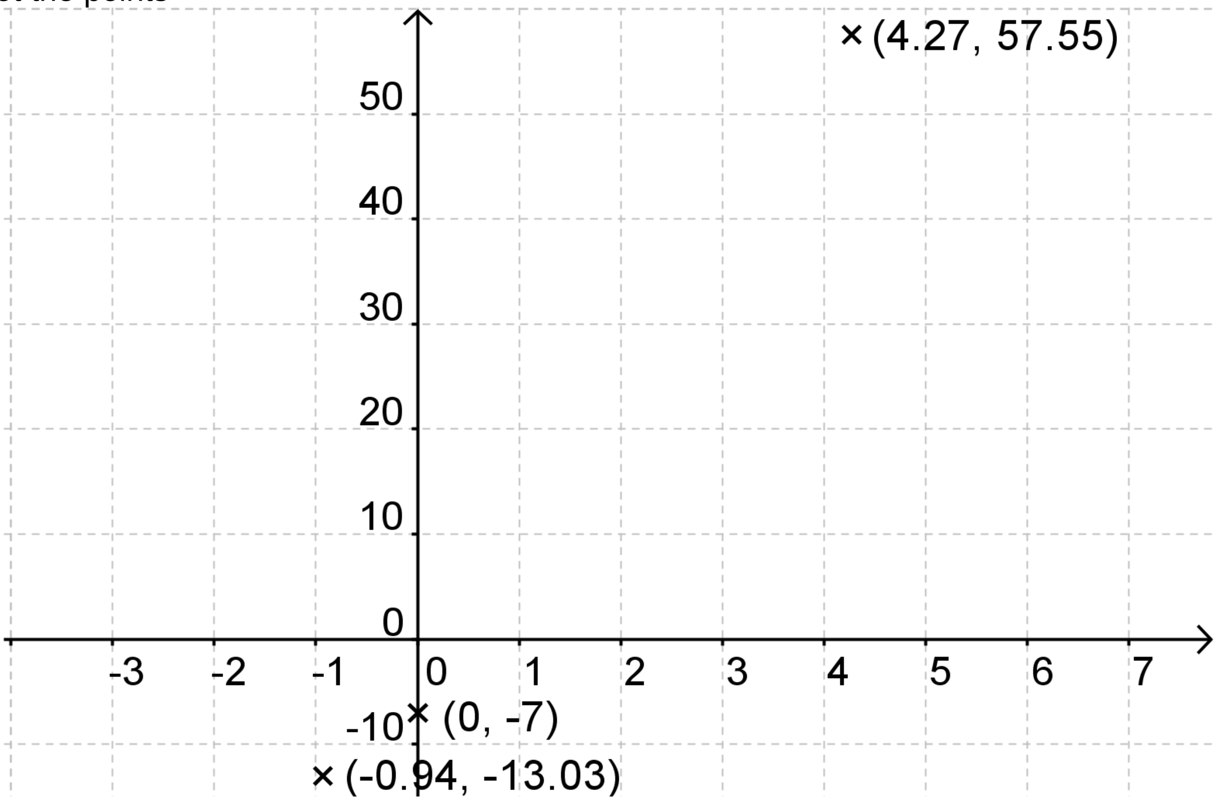
From the table you can see the shape of the curve.

The x co-ordinates of the turning points can now be found (use Eq 1)

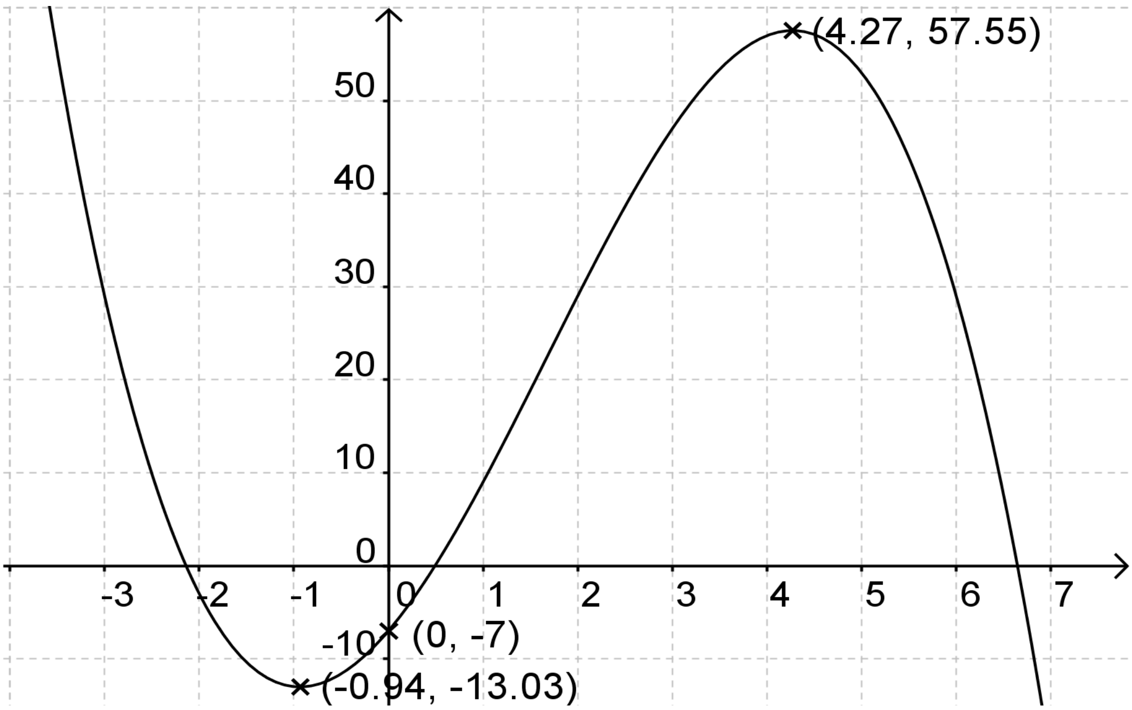
Local maximum is (4.270, 57 55) and the local minimum is (-0.9368, -13.03)

The graph can now be sketched

Plot the points



Mark all the major points that you know. It should now be possible to draw in the line (a smooth freehand curve)



Not bad. It looks very similar to the graph on page 1.

Another example - simpler this time

By using the Calculus find the maximum value of $y=3x-2x^2+7$

This example looks like a question from a quadratics assessment - (it is exactly the same thing!)

(Remember that the maximum or minimum is found by evaluating $x=\frac{-b}{2a}$ from $y=ax^2+bx+c$

$$y=ax^2+bx+c$$
$$\frac{dy}{dx}=2ax+b$$

At a local maximum or local minimum $\frac{dy}{dx}=0$ therefore

$$2ax+b=0$$
$$2ax=-b$$
$$x=\frac{-b}{2a}$$

In the example $y=3x-2x^2+7$

At a local maximum or local minimum $\frac{dy}{dx}=0$ therefore

$4x=3$ only one value - easy to calculate!

$$x=\frac{3}{4}=0.75$$

Show that $x=0.75$ does indeed give a maximum for y

	Left		Right
x	0.7400	0.7500	0.7600
$\frac{dy}{dx}$	+ 0.0400	0.0000	- 0.0400
Gradient	↗	→	↘

The maximum value of $y=3\times0.75-2\times0.75^2+7=8.125$

The procedure is as follows

1. *Write down the equation of the line.*

2. *Write down an appropriate statement relating the differential to the nature of the turning points.*
   For example "At a local maximum or local minimum $\frac{dy}{dx}=0$ *"*

3. *Differentiate the equation. (Find the differential of the equation) and set it equal to zero.*

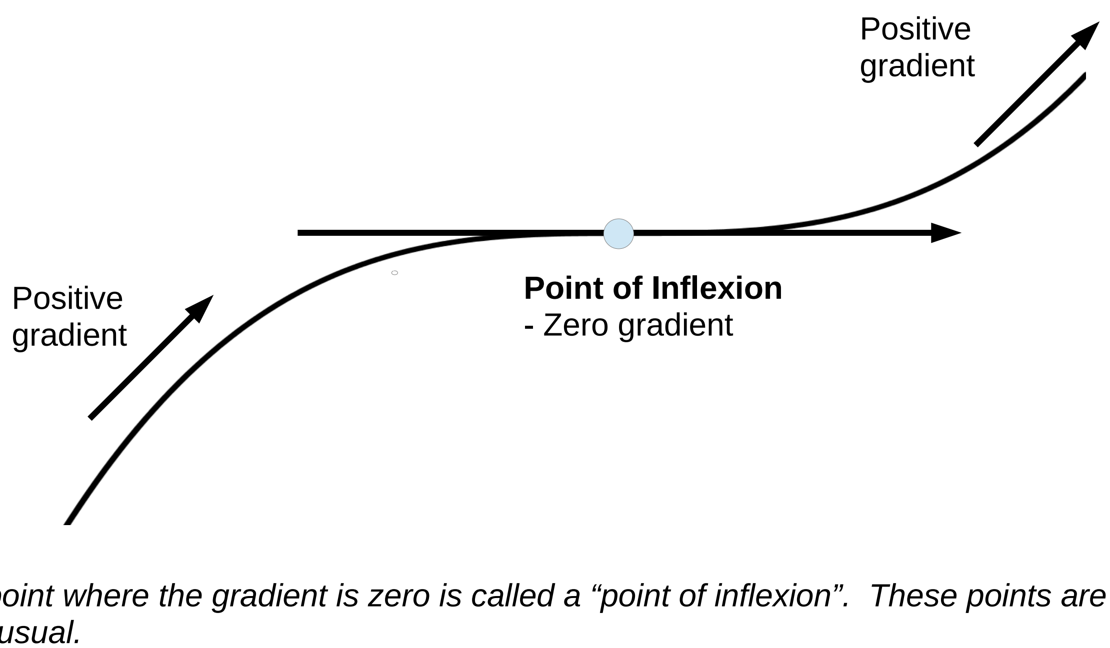
4. *Find the value(s) of the variable (* x *) that make the differential equal to zero.*

5. *Using a gradient table show that each value that you have found does, in fact, give local maximum or local minimum. (Use the differential)*

6. *Find the other co-ordinates (* y *)of the local maximum or local minimum.*

(7. *Sketch the graph and label it).*

Note: Although not covered in this course, maximum and minimum turning points are not the only points where the differential is equal to zero. Consider the graph below



The point where the gradient is zero is called a "point of inflexion". These points are not at all unusual.

Some questions for you to try.

Exercise 1

Find the maximum and/or minimum turning points and draw a labelled sketch of the graph. Label the graphs with the co-ordinates of the maximum/ minimum and the intercept of the graph with the vertical axis.

1. $y = x^3 - 6x^2 - 15x + 5$

2. $y = x^2 - 8x - 5$

3. $s = 12t^2 - 36t - t^3$ where s is distance in metres and t is time in seconds.

4. $h = 24 + 10t - 2t^2$ where h is height in metres and t is time in seconds.

5 $y = x^3 - 6x^2 + 9x - 2$

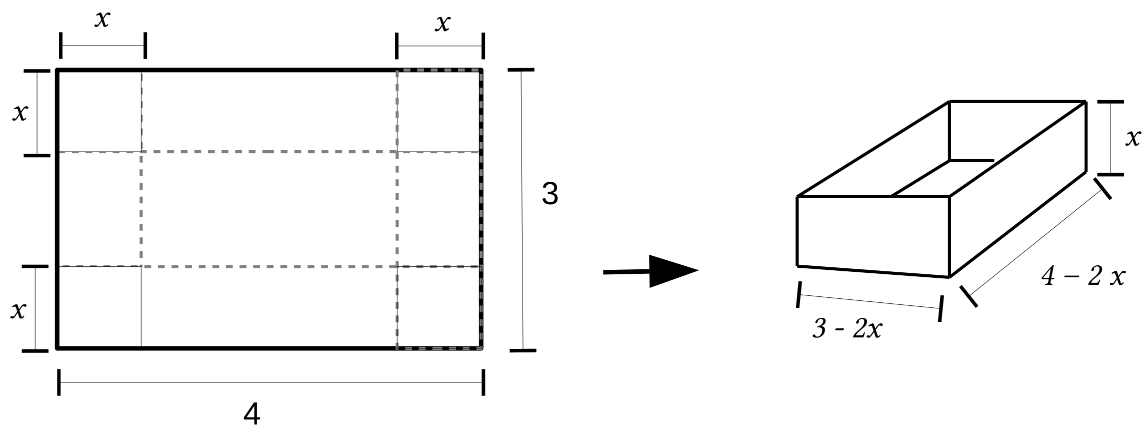
6. $v = 5h^3 + 4h^2 - 7h$

Practical Applications

Imagine you were given a stack of identical rectangular sheets of metal 3 metres by 4 metres and you were asked to carry out the following experiment.

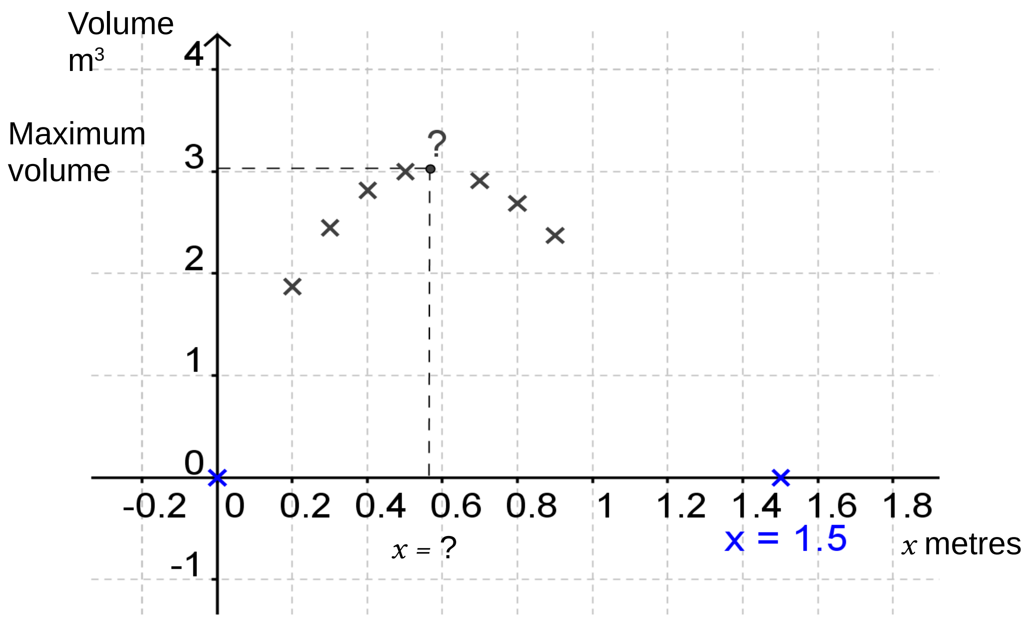
Make the largest volume open top box possible.

Cut along the solid lines and bend along the dashed lines. Weld the sides together.



This is a real world problem which could be solved satisfactorily by trial and error but it would take a long time and would cost a lot in terms of materials and labour.

If we did try to solve it by trial and error we could use a graph to help arrive at the best answer.
The graph shows that if you cut out nothing you get zero volume and if you cut out squares of side 1.5 you again get zero volume. A maximum volume must exist between these 2 points



We are now well on the way to setting up a mathematical model for the situation.

The graph is showing us a number of interesting things

1. There are 2 points where volume is zero - when $x=0$ and when $x=1.5$

1. If a line of best fit were drawn through the points there would be a value of x which would give a maximum volume. This would be a useful estimate and would be all that is required to make the box.

1. It should be possible to discover the exact value of x which gives the maximum volume by using maths instead of cutting, bending and welding a lot of metal.

Here's the maths.

The volume (V) of the box is

$V=(4-2x)(3-2x)x$ metres^3
(length times breadth times depth)

$$V=(12-8x-6x+4x^2)x$$
$$V=12x-14x^2+4x^3$$

At a maximum volume $\frac{dV}{dx}=0$

$$\frac{dV}{dx}=12-28x+4x^2$$

In this case, and **ONLY** because there is an x^2 term use the quadratic formula to find the values of x that will give $\frac{dy}{dx}=0$ (Zero gradient).

(where, $ax^2+bx+c=0$, $x=\frac{-b}{2a}\pm\frac{\sqrt{(b^2-4ac)}}{2a}$

$a=$

$b=$

$c=$

$x = 0.5657$ and $x = 1.768$ and (answers to 4 sig. fig.)

Which value of x gives the maximum volume?

I will set up a gradient table but, from the graph, it should be obvious that 1.768 cannot be a feasible value.

x	Left 0.5600	0.5657	Right 0.5700		*Left 1.7000*	*1.7680*	*Right 1.8000*
$\frac{dV}{dx}$	0.0830	0.0006 ≈ 0	-0.0610		*-0.9200*	*0.006* ≈ 0	*0.4800*
Gradient	↗	→	↘		↘	→	↗

Mathematical models often produce answers that do not work in the context of real life. It is important that you are able to identify the useful answers and be able to work with them.

We can now say that the maximum volume will be obtained if the value of x = 0.5657. We could have obtained a more precise answer but have to bear in mind the rest of the data and the requirements of the job.

The maximum volume is found by substituting for x for 0.5657 in the equation

$$V = 12x - 14x^2 + 4x^3$$
$$= 12 \times 0.5657 - 14 \times 0.5657^2 + 4 \times 0.5657^3$$
$$= 3.032\, m^3$$ (3032 litres)

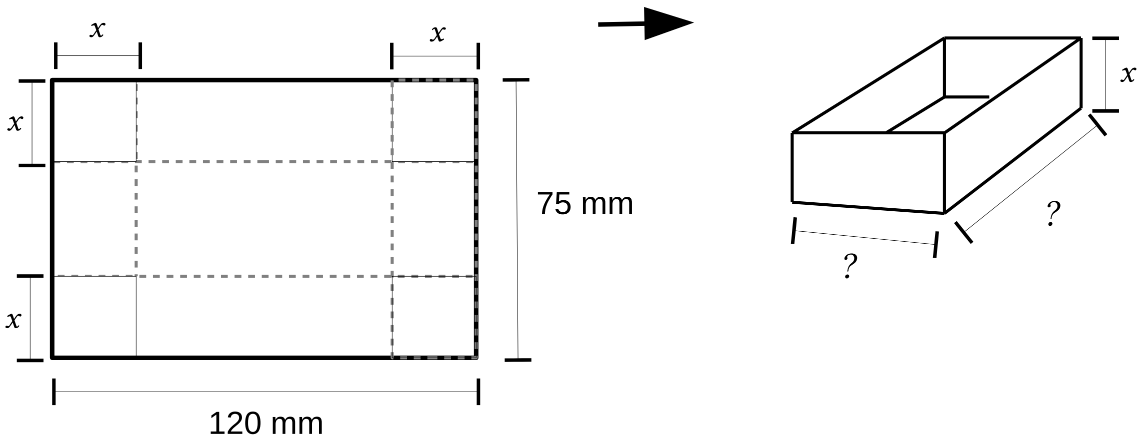
The maximum volume of an open top box, of the style shown, made from a sheet of metal 3 metres by 4 metres is 3.032 metres cubed.

Note: The answer that we have obtained, although it is correct, would be slightly different in practice because of the metal lost in bending and welding ..... But, of course, we could design a more complicated mathematical model to account for these factors.

Now some to try:

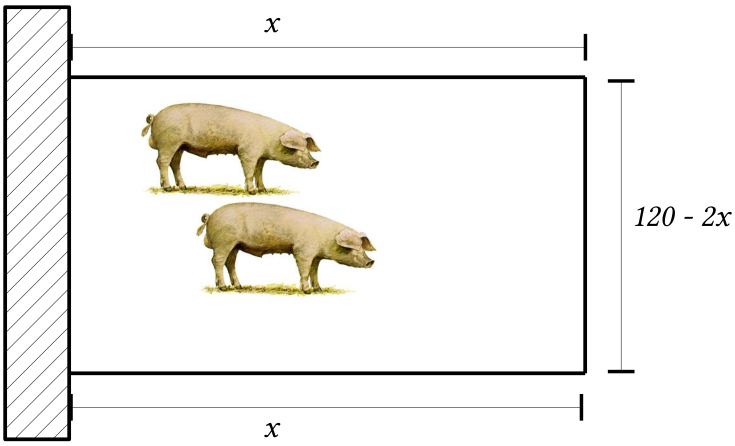
Exercise 2

1.



Calculate, using Calculus, the maximum volume of open top box that can be made from a sheet of metal measuring 120mm by 75mm. Determine the box dimensions.

2. A farmer has 120 metres of fencing which he wishes to use to hold a couple of pigs. The fence is only going to have to go round three sides as he is going to use an existing wall for the fourth side as shown below.

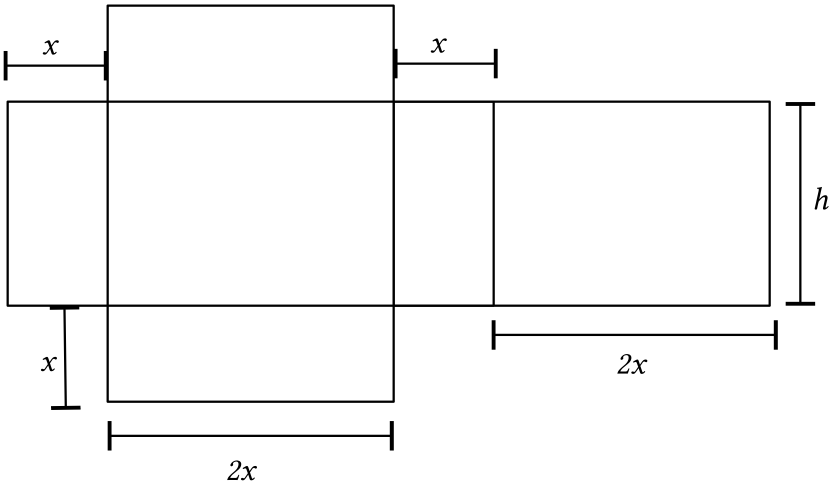


Find the maximum possible **rectangular** area that he can enclose and the length of the sides of the enclosure.

(Is there a better shape that would give the pigs more area?)

3. A closed box is to have a volume of $72\ m^3$.

Find the dimensions of the box if the width is to be twice the length and the surface area is to be a minimum.



Volume $V = 2x \times x \times h = 72$ therefore $h = \frac{72}{2x^2} = \frac{36}{x^2}$

Surface Area $= (x + 2x + x + 2x) \times h + 2 \times 2x \times x$

sub for h $= 6x \times \frac{36}{x^2} + 4x^2$

$$A = \frac{216}{x} + 4x^2$$

4. A company manufactures electric drills and estimates that the average cost of producing x drills per month is given by the equation

$$c(x) = 0.008x + 0.02 + \frac{8000}{x}.$$

Find the number of drills that will have to be made to minimise the average cost.

5. A flywheel rotates through an angle of θ (radians) according to the formula

$\theta = 5 + 14t - 2t^2$ where t is the time in seconds.

Show that the wheel slows down for the first 3.5 seconds and then changes its direction of rotation.

6. A projectile is fired from 3 m above ground level with equation of motion of

$s = 150t - 10t^2 + 3$ metres t in seconds.

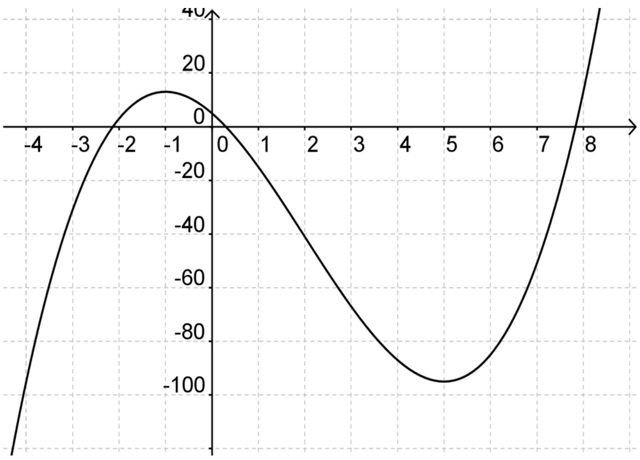
Use differentiation to help you calculate the time it reaches maximum height and also calculate the height. You will need a gradient table

Answers:

Note. In every answer you must have made a statement to the effect that at a maximum or minimum the differential is equal to zero AND you must have demonstrated that you have found maximum and/or minimum values by showing this in a gradient table or by other means.

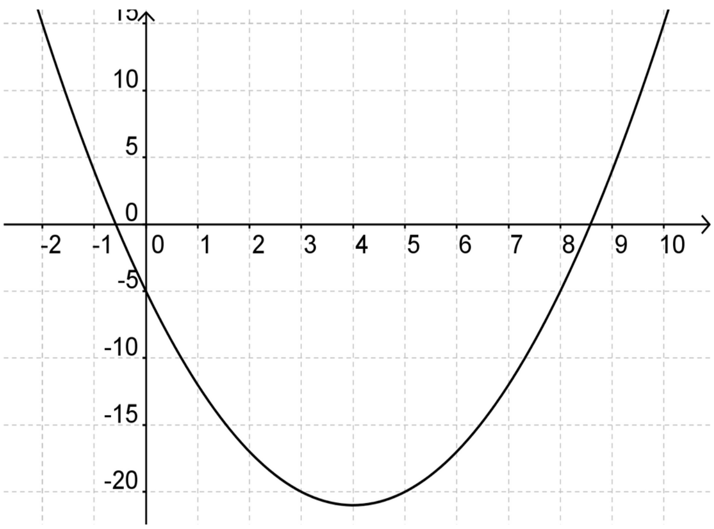
Exercise 1

1.



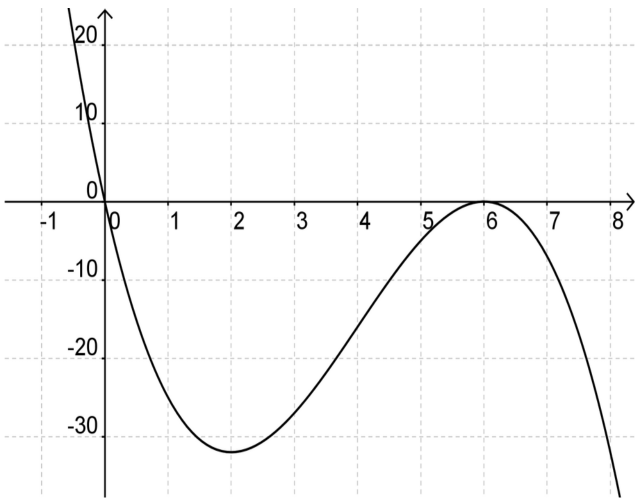
Maximum at (-1,13) and a minimum at (5,-95). Crosses the y axis at (0,5)

2.



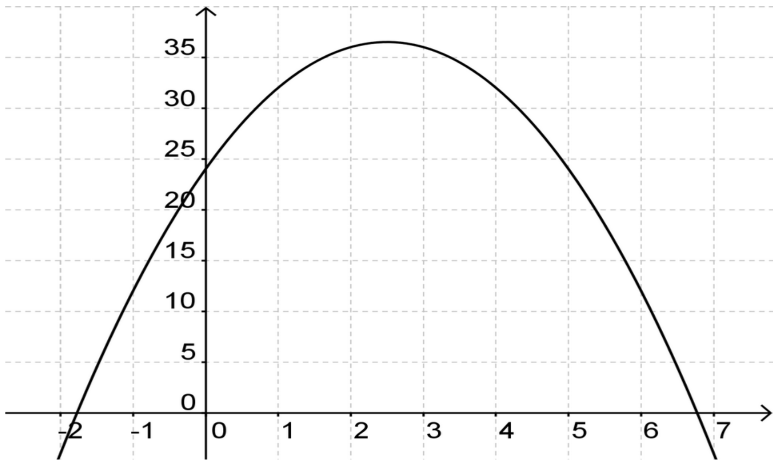
Minimum at (4,-21) . Crosses the y axis at (0,-5)

3.



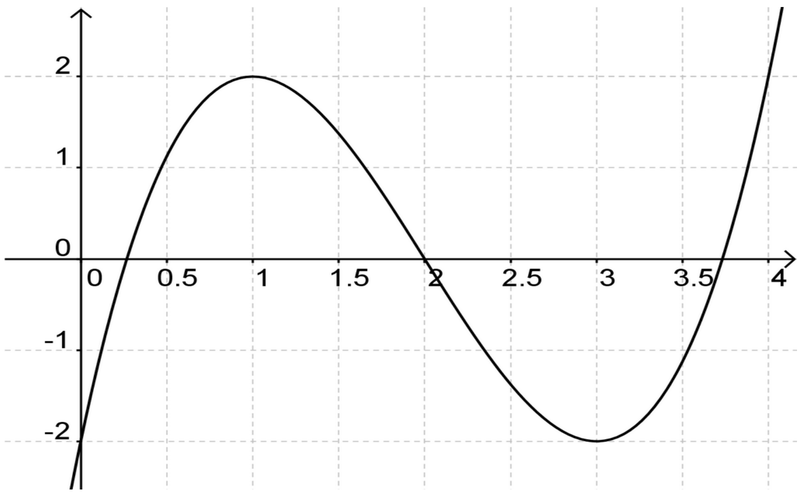
Maximum at (6, 0) and minimum at (2, -32). Crosses the y axis at (0,0)

4.



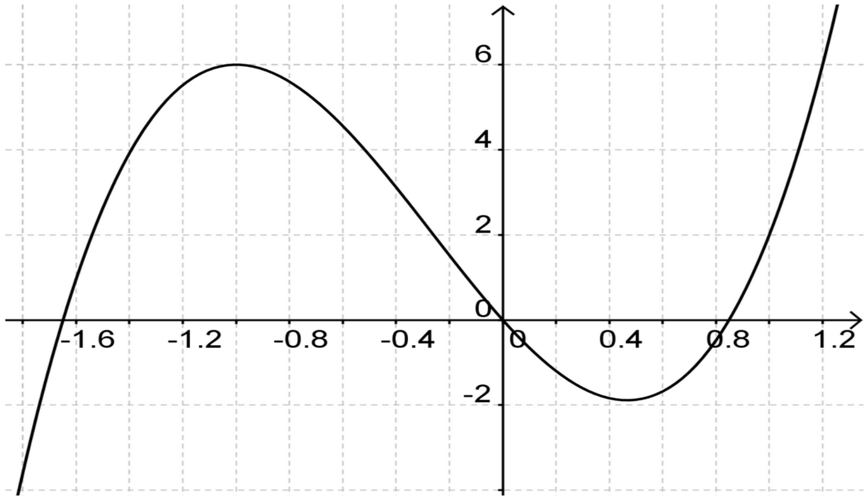
Maximum at (2.5, 36.5). Crosses the y axis at (0, 24)

5.



Maximum at (1, 2) and a minimum at (3, -2). Crosses the y axis at (0.-2).

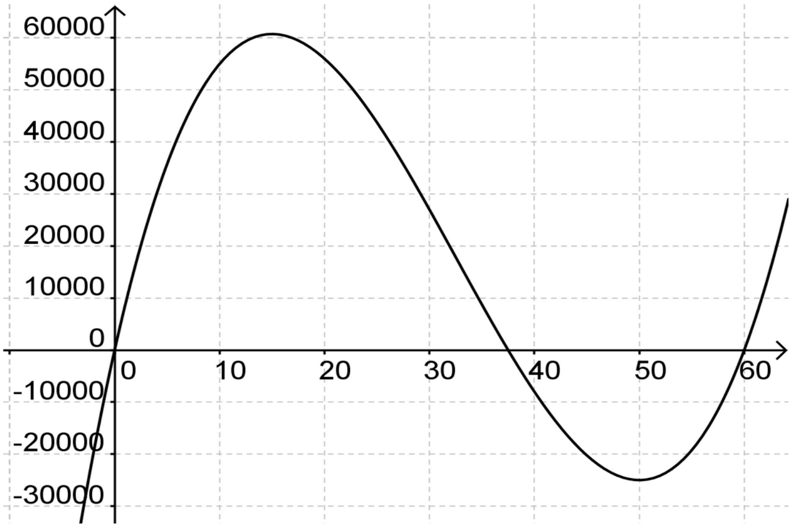
6.



Maximum at (-1, 6) and a minimum at (0.467, -1.887). Crosses the y axis at (0, 0)

Exercise 2 - Graphs are not required - they are here for interest only.

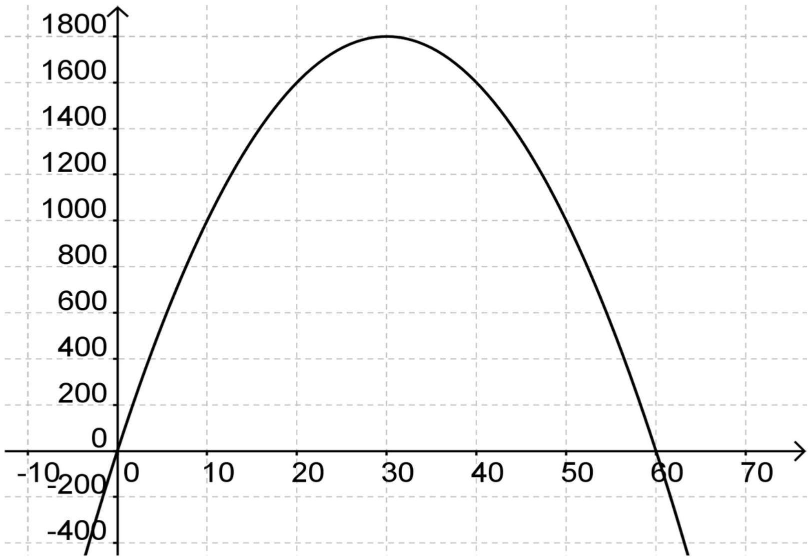
1.



Maximum at (15, 60750) *and a minimum at (50, -25000)*

Maximum volume is 60750 mm^3 or .
Box dimensions are 90 mm X 45 mm X 15 mm

2.

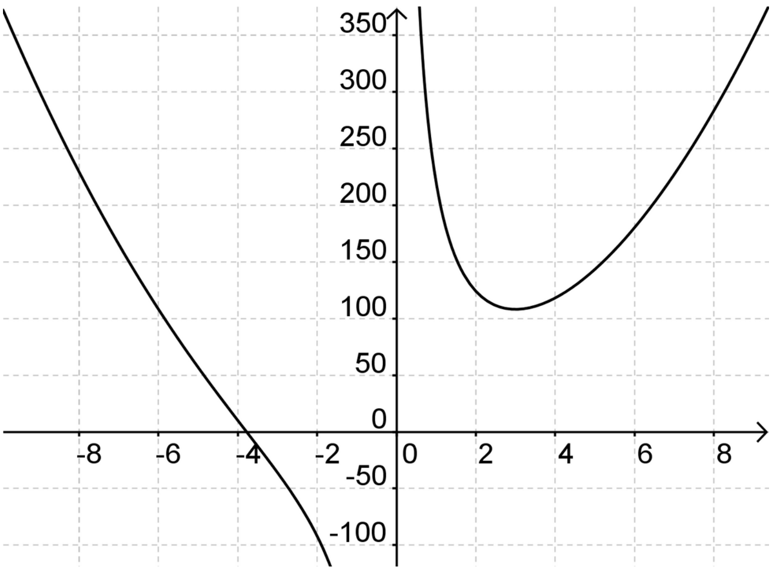


Maximum at (30, 1800).

Maximum area is 1800 m^2. Dimensions of fence are 30 X 30 X 60 metres.

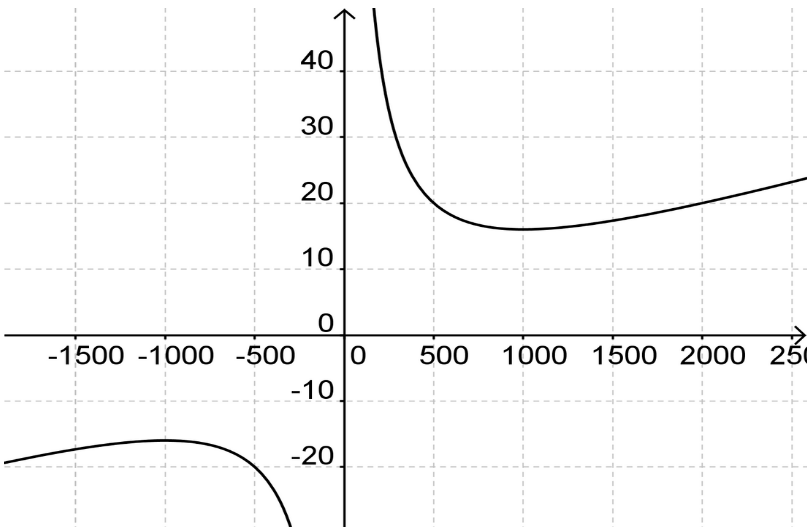
(A semicircle would enclose an area of 2292 m^2 approx.)

3.



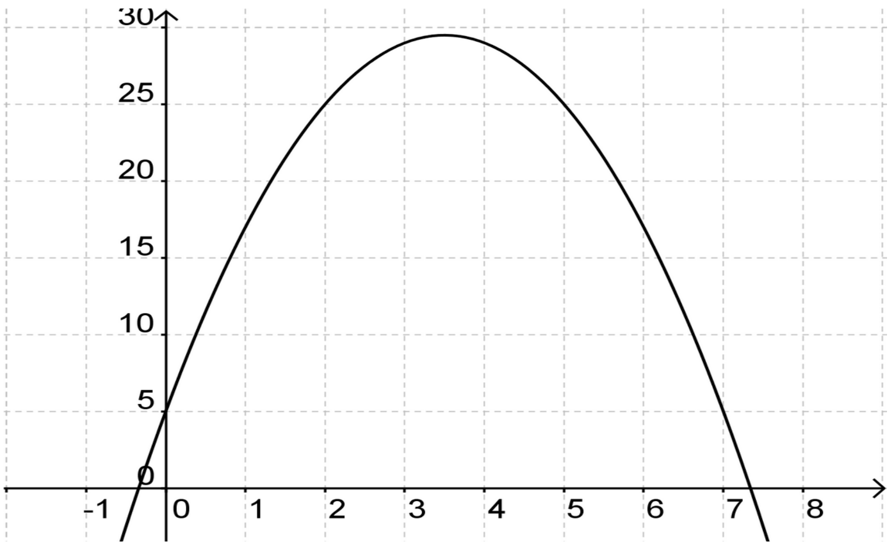
Minimum is at (3, 108).
Minimum surface area is 108 m^2.
Dimensions of the box are 6 m X 3 m X 4 m.

4.



Minimum is at (1000, 16.02) *and maximum is at (-1000, -15.98)*
In order to minimise the average cost of production per drill 1000 drills have to be made.
(average cost of production is £16.02 per drill)

5.

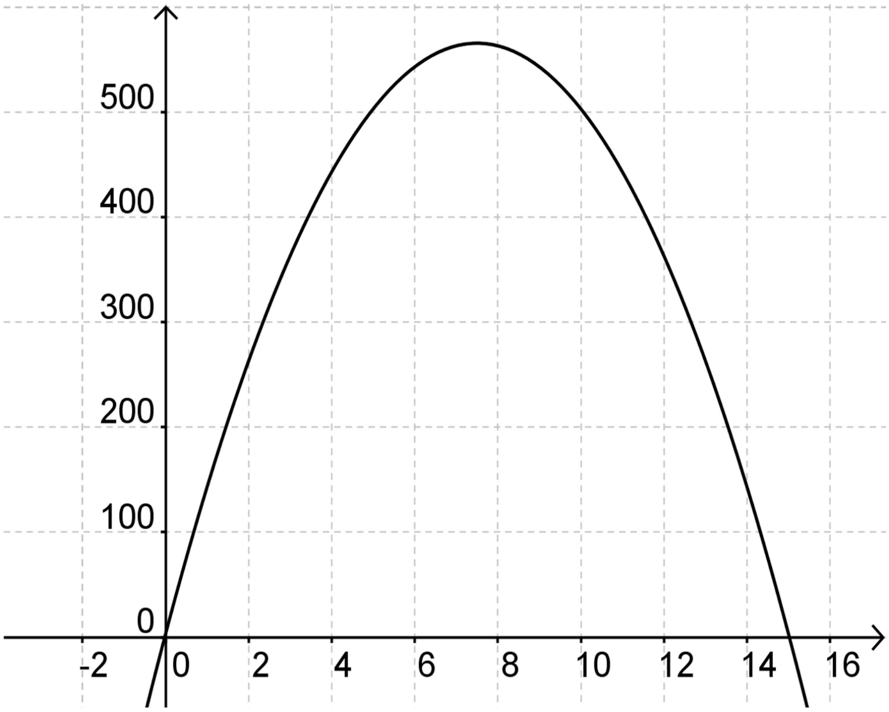


Maximum at (3.5, 29.5)

Show, from the gradient table, that the velocity is positive to the left of $t=3.5$ but is zero at $t=3.5$ therefore the velocity is decreasing until $t=3.5$ i.e. The wheel is decelerating, or slowing down, until $t=3.5$.
At $t=3.5$ velocity is zero - the wheel has stopped.
After $t=3.5$ velocity is negative and getting more negative the as time passes. As the velocity is negative it must be moving in the opposite direction from which it started.

(Velocity has a magnitude (how much) and a sign (+ or -) which indicates the direction of movement.)

6.



The projectile reaches a maximum of 565.5 m at 7.5 s.
Note that the quadratic formula was NOT required (because it is not needed and will not work (dividing by 0)).